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Imitation games: Power-law sensitivity to initial conditions and nonextensivity

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We exhibit, at the self-organized critical state, a power-law sensitivity to the initial conditions in the system of competing logistic maps introduced by Suzuki and Kaneko [Physica D **75**, 328 (1994)], which modelizes the battle of birds defending their territories. From the associated exponent we obtain the value for the entropic index q, which defines the recently introduced nonextensive generalization of the Boltzmann-Gibbs thermostatistics. In addition, we calculate the dynamical exponent z. We obtained $q \simeq -0.69$ and $z \simeq 1.32$ for the mean-field-type calculation ($d=\infty$) and $q \simeq -0.39$ and $z \simeq 1.12$ for a square lattice ($d=2$) model. Finally, we have generalized Suzuki and Kaneko's model, using an extended form of the logistic map, and have calculated the corresponding values of q for $d=1,2,\infty$.
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I. INTRODUCTION

Self-organized criticality is an interesting and quite ubiquitous phenomenon. Indeed, it is experimentally observed and theoretically studied in a wide variety of physical systems, which include sandpiles, ricepiles, and earthquakes [1]. It has also been observed in the model introduced by Suzuki and Kaneko [2], who proposed a set of coupled logistic maps to mimic the singing imitation games that constitute the basis of the battle for territory defense in various species of birds. This model has successfully described a variety of biologically relevant features. However, its sensitivity to the initial conditions has never been focused on up to now, to the best of our knowledge. This property, which essentially characterizes chaotic behavior, is herein studied quantitatively. Moreover, we show that it is intimately related to thermostatistical nonextensivity in the sense we now describe.

It has been known for many years [3] that standard statistical mechanics fails to describe pathological systems such as those that include long-range interactions (as well as long-range microscopic memory or fractal boundary conditions in space-time, among others). To discuss this type of anomalous system, one of us proposed [4] a thermostatistics based on the generalized entropic form

$$S_q = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}, \tag{1}$$

where W is the total number of configurations, $\{p_i\}$ are the associated probabilities, k is some suitable positive constant, and $q \in \mathcal{R}$ is the index that allows for the generalization. It is straightforwardly verified that, in the $q \to 1$ limit, Eq. (1) reduces [using $p_i^{q-1} \sim 1 + (q-1)\ln p_i$] to the well-known expression

$$S_1 = -k_B \sum_{i=1}^{W} p_i \ln p_i . \tag{2}$$

Let us mention that if we consider a system composed of two *independent* subsystems A and B (in the sense that the composed probabilities *factorize* into those of A and B), we easily verify that $S_q(A+B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B)$, which clearly exhibits *nonextensivity* if $q \neq 1$.

The above generalization has been applied to a wide variety of physical situations including self-gravitational systems [5,6], turbulence in pure-electron plasma [6], the dynamic linear response for nonextensive systems [7], Lévy-like [8] and correlatedlike [9] anomalous diffusions, the solar neutrino problem [10], cosmology [11], long-range fluid and magnetic systems [12], and optimization techniques [13].

This formalism has been shown [14,15] to be connected to the sensitivity of nonlinear dynamical systems. More precisely, the numerical analysis of logisticlike maps as well as of the Bak-Sneppen model for biological evolution has shown that, whenever quantities such as the Lyapunov exponent vanish (e.g., at the onset of chaos), the standard, exponential type of sensitivity to the initial conditions is often replaced by a weaker, power-law type of sensitivity. It has been argued that, within the nonextensive formalism associated with Eq. (1), the sensitivity to the initial conditions of possibly large classes of nonlinear maps is [14,16], e.g., for a one-dimensional map,

$$\xi(t) \equiv \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} = [1 + (1-q)\lambda_q t]^{1/(1-q)}, \tag{3}$$

which is the exact solution of the differential equation $d\xi/dt = \lambda_q \xi^q$. We see that (i) for $q=1$ we recover the traditional exponential behavior $\xi(t) = e^{\lambda_1 t}$ and (ii) if $t \to \infty$, $\xi \propto t^{1/(1-q)}$, which corresponds to the so-called weak chaos. The Pesin equality states that, under some conditions, the Kolmogorov-Sinai entropy K_1 (where the subindex 1 refers to the traditional statistics) equals the Lyapunov exponent λ_1. In [14], both magnitudes, namely, the Kolmogorov-Sinai

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entropy and the Lyapunov exponent, were independently generalized to K_q and λ_q, respectively [here q precisely is the entropic parameter appearing in Eq. (1), on which the definition of K_q is based]. In the $q \rightarrow 1$ limit K_q and λ_q reduce to K_1 and λ_1, respectively. It was shown that K_q coincides with λ_q, thus generalizing, for arbitrary q, the Pesin equality as

$$K_q = \lambda_q \quad (\lambda_q \geqslant 0). \tag{4}$$

Let us stress that since it is from the generalized entropy S_q that we have obtained K_q and since Eq. (4) exhibits the equality between K_q and λ_q, when we focus on λ_q in some context we are referring also to the generalized entropy and in particular to a specific value of q.

In the present work, we show how these ideas apply to Suzuki and Kaneko's imitation games [2]. After some transient, this model spontaneously evolves towards a critical state located at the edge between chaos and periodic windows, i.e., in a parameter region where the Lyapunov exponent of individual maps is close to zero. This peculiar steady-state-like behavior is sometimes referred to as *intermittent chaos* or *weakly chaotic.*

II. IMITATION GAMES

Motivated by the observed complexity of bird songs and by the observation that birds try to imitate each other's songs for defense of territory, Suzuki and Kaneko advanced a model to study mutual imitation games among artificial birds. They confirmed, employing a set of interacting logistic maps, the evolution to the edge between chaos and windows.

Let us briefly review their model. We use, for the imitation games, the logistic maps

$$x_{n+1}(i) = 1 - a(i) x_n^2(i) \tag{5}$$

that play by pairs (i,j) $(x_n(i) \in [-1,1];\ a(i) \in [0,2];\ i = 1,2,\ldots,N)$. Initially, each map starts with random values for both $x_0(i)$ and $a(i)$ and repeats its dynamics for a time T_{rel} (rel stands for relaxation) long enough to approach its individual attractor. Then, during a time T_{imi} (imi stands for imitation), the ith map (for all values of i) modifies its dynamics (imitates) with a feedback from the jth map (we describe later on how the sequence for j is chosen)

$$x_{n+1}(i) = 1 - a(i)\{[1-\epsilon(i)]x_n(i) + \epsilon(i) x_n(j)\}^2, \tag{6}$$

where the $\epsilon(i)$ are the parameters (random numbers uniformly distributed between zero and one and chosen once forever) that characterize how strong the imitation is. After this time, the ith map returns to its own dynamics [this is to say, it runs with Eq. (3) starting from its current value for x_n] during a period T_D. During time T_D a quantity $D(i,j)$ is calculated that measures the distance between the imitated values $x_n(i)$ and the original ones $x_n(j)$:

$$D(i,j) = \sum_{n=T_{rel}+T_{imi}+1}^{T_{rel}+T_{imi}+T_D} |x_n(i) - x_n(j)|^2. \tag{7}$$

For the present work, and following Ref. [13], we adopt $T_{rel} = T_{imi} = T_D = 30$.

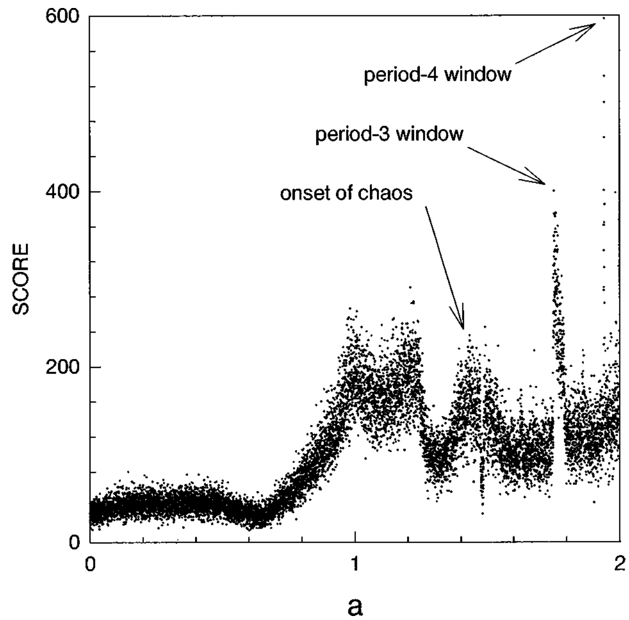

FIG. 1. Stationary score versus a for 100 logistic maps playing all with all (mean-field-like model). The sum was calculated during $\sim 10^8$ time steps after a long relaxation period. One time step is a single two-bird game. The period-three window, period-four window, and onset of chaos are pointed out by arrows; other remarkable peaks correspond to other windows and bifurcations.

By inverting the role of the two maps and repeating the same procedure, the quantity $D(j,i)$ is calculated. If $D(i,j) < D(j,i)$, then we will say that the ith map imitates better the jth map than the other way around; hence the ith map "wins." By "wins" we mean that the parameter $a(j)$ is substituted by a value in the vicinity of $a(i)$ or, more precisely, it becomes $a(i)+\delta$, where δ is a small random number, representing a *mutational* possibility; following [2], δ is extracted from the Lorentzian distribution $P(\delta) = \mu/(\mu^2 + \delta^2)$ with $\mu = 0.001$. If $D(i,j) > D(j,i)$, the "winner" now is the jth map and we proceed as before, i.e., we change the parameters of the ith map. We have not used homogeneous random distributions to determine δ because, as noted in [2] and confirmed by us, this can lead to parameters a that become trapped at intermediate, meaningless values. Let us add that no further variations were performed on the ϵ-parameter distribution because, as noted by Susuki and Kaneko [2] and confirmed by our own simulations, the distribution of this parameter appears to be essentially "irrelevant."

In order to determine the values of a that often win and following the lines of [2], we define a score for each value of a. We increase by one a counter associated with the value of a of the winning map and by zero the counter corresponding to the loser. In Fig. 1 we show the score as a function of a. Note the importance of the period-three and period-four windows. Figure 2 shows the score as a function of the logistic-map Lyapunov exponent λ. Note now the peak on the zero value. Let us recall that, for a single logistic map, $\lambda = 0$ precisely corresponds to $q \neq 1$, i.e., to a power-law (instead of exponential) dependence of the sensitivity to the initial conditions.

III. RESULTS

The Hamming distance between two systems (the *original* and its *replica*) of N maps each, at any time n, is defined as

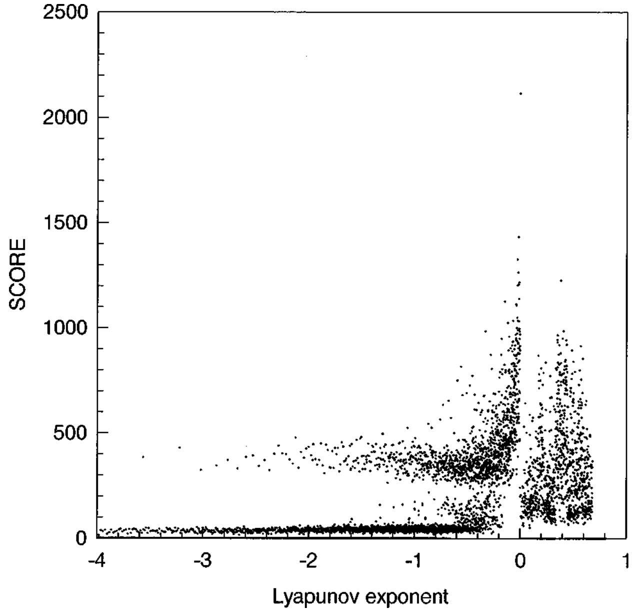

FIG. 2. Stationary score versus Lyapunov exponent (λ) as a result of the calculation in Fig. 1. Note the peak around $\lambda=0$, it used a bin size of 0.0001.

$$H_n=\sum_{i=1}^{N}\frac{|a_n(i)-a_n'(i)|}{N}, \tag{8}$$

where the prime stands for the replica sample. Note that the Hamming distance is calculated over the parameters a of the maps and not over the entire phase space (which includes the x variables) because it is the a (and not the x) that self-organize. In our simulations, we have let a system of N maps ($N=125,250,500,1000$) relax and then we have constructed a replica by randomly changing (in ± 0.001) the parameters a of (typically) ten randomly chosen maps. After this is done, we calculate the ''initial'' damage or Hamming distance H_0. It is clear that, for increasingly large values of N and since we have fixed the number of different maps in the replica to ten, the Hamming distance tends to zero. By so doing we numerically approach the definition of the Lyapunov exponent, which demands a vanishingly small initial discrepancy between the replicas. It plays here an analogous roll to the one played by Lyapunov exponents for individual maps. One reason that we used the Hamming distance for characterization of the system is of practical order: The search for the largest eigenvalue for the set of maps would imply the diagonalization of a huge matrix an enormous number of times. This task is well beyond the current possibilities of computational media.

By allowing both the original and the replica systems to follow the previously described dynamics, *with identical sequences of random numbers* (in accordance with the standard spreading-of-damage procedure), we calculated the ratio H_n/H_0. We use as the unit of time (n is increased by one) each single game between two birds. In Fig. 3 we show, for the mean-field-like model (every bird plays with each one of the others), the log-log time dependence of the average (over 50 realizations) of H_n/H_0 for various system sizes. It is a general feature that H_n/H_0 grows with time following a power law (basically the same for all sizes) up to a size-dependent point, after which a stationary plateau is reached. The slope of the first part of the curves is 0.59 ± 0.02. Since this value equals [14] $1/(1-q)$, we determine $q=-0.69\pm0.02$. For the $d=2$ square lattice model (with periodic boundary conditions) we have obtained $q=-0.39\pm0.02$. These values can be compared with those associated with other models, namely, the logistic map at its threshold to chaos ($q=0.4$ [14]), the Bak-Sneppen model for biological evolution at the self-organized critical state ($q=-2.1$ [15]), and the rice-pile model ($q=-0.12$ [17]).

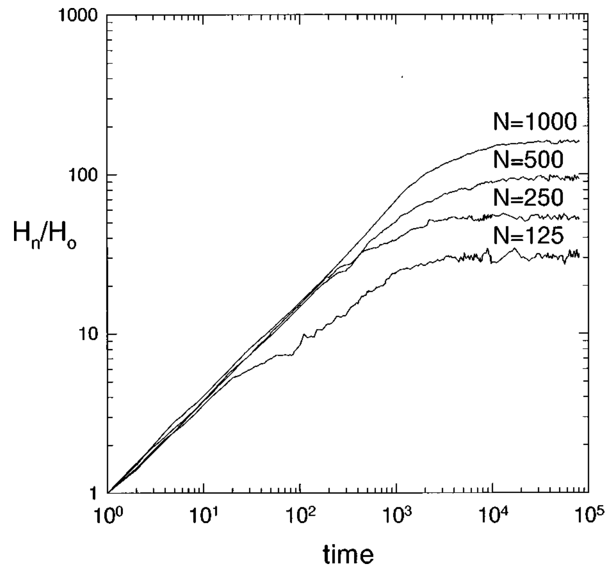

FIG. 3. A log-log plot of the Hamming distance versus time for various system sizes. All the curves coincide rather well on the straight-line part. The time step is a game between two maps.

The dynamical exponent z is usually defined through $\tau\sim L^z$, where τ is the time during which the system behaves dynamically and L is the linear size of the system. More precisely, the time τ is defined as the time needed for the damage to reach, except for statistical fluctuations, its stationary value. So τ was estimated through the intersection, in Fig. 3, of the two straight lines respectively defined by the plateaulike stationary value and the power-law increasing regime. It was found (see Fig. 4) that the dynamical exponent $z=1.32\pm0.01$. For the $d=2$ square lattice model we obtained $z=1.12\pm0.02$. These values can be compared with those obtained for the Bak-Sneppen model ($z=1.56$ [15]), the rice-pile model ($z=1.3$ [17]), and the square lattice Ising ferromagnet ($z=2.16$ [18]).

In order to explore the dependence of the parameter q upon the nature of the maps, we implemented the algorithm for the logisticlike family of maps

$$x_{n+1}(i)=1-a(i)|x_n(i)|^{\zeta} \tag{9}$$

in dimension $d=1,2,\infty$. The explored values of ζ belong to the interval $[2,\infty]$. In Fig. 5 we present the dependence of q on (ζ,d). The tendency of q to the value 1 (extensivity) as $\zeta\to\infty$ is apparent, *for all dimensions*, whereas for all finite values of ζ, q does depend on dimensionality. There is no threshold for the transition from nonextensivity to extensivity; the change is gradual and only in the $\zeta\to\infty$ limit is extensivity recovered.

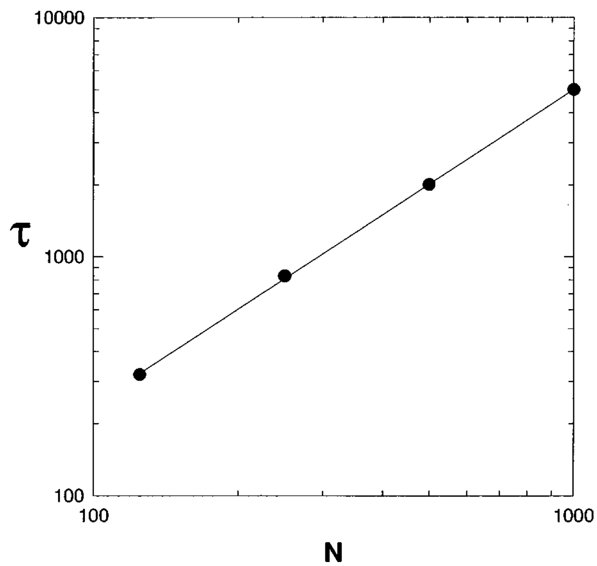

FIG. 4. A log-log plot of time τ needed to reach the ''knee'' (in Fig. 3) versus system size. The slope of the straight line is 1.32 with good accuracy. The time units are the same as in Fig. 3.

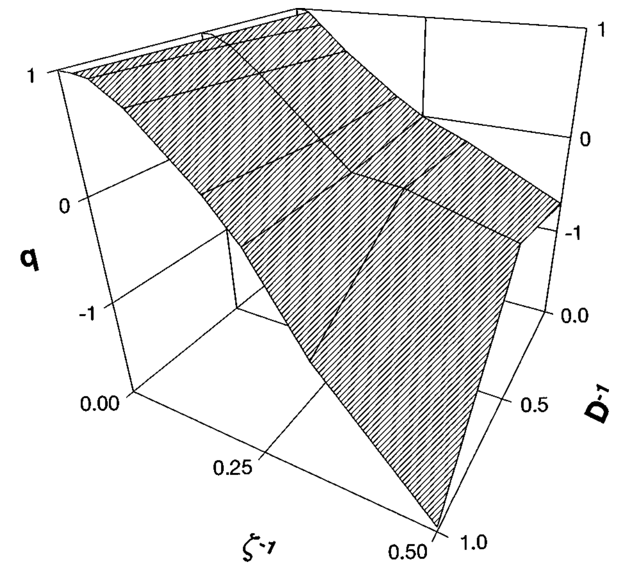

FIG. 5. Dependence of q on the dimension of the model and on ζ. The calculations were carried for $d=1$, 2, and ∞ (we recall that $d=\infty$ corresponds to the mean-field approximation) and for ζ = 2, 3, 4, 5, 10, 20, and 40. In the $\zeta \to \infty$ limit, $q \to 1$ for all values of d.

In addition to the above results, we have checked the sensitivity to the initial conditions of the set of values $\{x\}$. After a simple transient, a rather trivial diffusivelike behavior is observed, which reconfirms that, in the present problem, it is in the space of the $\{a\}$ that the nontrivial results are conveniently revealed.

IV. CONCLUSIONS

We have obtained for a set of imitating logistic maps a power-law increase of the Hamming distance with time. For finite systems a plateau is observed if enough time elapses, but, in the limit $N \to \infty$, the nontrivial power-law regime should last forever. As the number of elements in the system increases so does the time required to reach the plateau; from this dependence the dynamical exponent z was obtained.

As seen from the study with logisticlike maps in several dimensions d and with different ζ exponents, the result is robust in the sense that a power-law increase of the Hamming distance with time is always obtained. There is a dependence of q on both the dimension of the model and the exponent ζ of the maps, pointing to different, continuously varying, universality classes. In particular, the limit $q=1$ (extensivity) is reached, for all dimensions, when $\zeta \to \infty$.

Summarizing, we have studied, at the self-organized critical state, the sensitivity to initial conditions of Suzuki and Kaneko's model for imitation games between birds. We have exhibited nontrivial power-law behaviors, which enable, among others, the connection with nonexistence statistics. In other words, we have illustrated herein how the entropic exponent q can be derived from the knowledge of the microscopic dynamics. This appears to be an important step within nonlinear dynamical systems such as those exhibiting self-organized criticality. Indeed, the present work suggests a manner of predicting fundamental properties of not necessarily extensive systems from their very microscopic mechanisms, which determine, through the value of q, the degree of nonextensivity.

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